MATHEMATICS IN THIS UNIT
Multiplication Patterns

Taken from the *Math Trailblazers* digital Teacher Guide

Multiplication Facts. In this unit we focus on multiplication facts. Students engage in activities that will help them develop strategies for learning the facts. We expect students to have a variety of strategies for dealing with the facts in third grade and to achieve fluency by the end of the year. This unit introduces the array model of multiplication, investigates multiplication and division with zero, and explores the patterns found when multiplying by multiples of 10 and 100.

Research has shown that fact retention is higher when facts are learned in a meaningful way. As students investigate patterns among the multiplication facts and use them in problem-solving situations and games, they will develop the ability to quickly recall them when needed.

"Fluency with the basic number combinations develops from well-understood meanings for the four operations and from a focus on thinking strategies" (National Council of Teachers of Mathematics, p. 152). Fluency with basic procedures enhances conceptual understanding of new material.

Strategies for Learning Facts. "Children who understand the big ideas of composition and decomposition are more likely to invent reasoning strategies, such as translating combinations into easier or known expressions" (Baroody, 2006).

The sequence for practicing and assessing the multiplication facts is based on studying groups of facts that are organized around reasoning strategies. In this unit, students build the multiplication table using these strategies in Lessons 3, 4 and 5.

- In Lesson 3, students begin by filling in the columns for the 0, 1, 2, 3, 5, and 10 using skip counting and other patterns (e.g., any number times 0 is 0).
- In Lesson 4, students build rectangular arrays and observe that the order of the factors does not matter ($3 \times 5 = 5 \times 3$). Using this information, they fill in the rows corresponding to the columns for 0, 1, 2, 3, 5, and 10. They build square arrays of different sizes, and using the total number of squares in the arrays add the square numbers to the table. At this point the students have completed the table as shown in Figure 1. The facts on the table are usually relatively easy for students to learn, and they can see that there are not many more facts left.

	☆	☆	☆	☆		☆					☆
×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20					40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18		30	36				60
7	0	7	14	21		35		49			70
8	0	8	16	24		40			64		80
9	0	9	18	27		45				81	90
10	0	10	20	30	40	50	60	70	80	90	100

Figure 1: *Sample multiplication table*

- In Lesson 5, they first develop strategies for finding the nines, add them to the table, and then discuss patterns for these products. At this point all but six facts (and their turn-around facts) are on the table. Students then complete the table with these "last six facts" using various strategies. One strategy students use is doubling, e.g., 4×7 is double 2×7 which is double 14 or 28. They learn to break apart products into easier facts. For example to solve 7×8, they break 7 into 2 and 5 and add 2×8 and 5×8 to get $16 + 40 = 56$. See Figure 2

Students focus on one group of facts in each unit from Unit 8 through Unit 13. The groups are based on the sequence described above. Students practice the fact groups in this order to help them learn the handy facts first and then use those handy facts to develop strategies for the more difficult facts. Figure 3 shows the distribution of fact practice and assessment throughout the remaining units:

Unit	Multiplication Facts Group	Focus
8	5s and 10s	Use strategies fluently
9	2s and 3s	
10	Square Numbers	
11	Nines	
12	Last Six Facts	
13	Last Six Facts	

Figure 3: *Sequence of the study of the multiplication facts in Units 8–13*

For more information on the use of strategies to learn the multiplication facts, see the TIMS Tutor: Math Facts in the *Professional Development Resource*.

Making Strategies Explicit. Using a problem-based approach and focusing on reasoning strategies are just as important, if not more so, for developing mastery of multiplication and related division facts. Many students will develop their own strategies for solving multiplication facts using their number sense, based on the addition facts they know, and based on their ability to skip count. Some students take more time to develop strategies and require explicit models to develop more efficient strategies. There is not a "best" strategy for a multiplication fact but rather students should be encouraged to develop a variety of flexible strategies that work for them. Special attention should be given to modeling strategies that encourage students to compose and decompose numbers and that identify and use patterns.

Effective Practice. Practice is an essential part of developing fluency with the basic facts. Students practice the facts as they take part in activities, labs, and games as they learn new content (NRC, 2001, p. 352). In this unit, students are introduced to the *Triangle Flash Cards* to practice their multiplication facts for the 5s and 10s. Students will use these cards as they focus on each group of facts. These same cards will be used in fourth grade to practice division facts. By using these cards to practice the basic facts, students strengthen their understanding of multiplicative reasoning, e.g., the relationships between factors and multiples and between multiplication and division.

Systematic practice and assessment of the facts occurs in the Home Practice and Daily Practice and Problems in each unit. See Figure 3 for the sequence. Examples of practice for the 5s and 10s, the designated fact group for Unit 8, are shown in parentheses.

- In each unit, students are asked to assess their knowledge of the facts using *Triangle Flash Cards* and to practice those that they need to study. (In this unit, students practice and assess 5s and 10s using the flash cards in Lesson 5.)
- Some DPP items practice the facts using word problems or other contexts such as measurement or geometry. (DPP items L, P, R, X, Z, FF)
- DPP items may encourage the use and discussion of specific concepts as students practice the facts. (DPP items H, M, N)
- Students use multiplication facts to complete open number sentences. (DPP items Y, BB, II, JJ)
- Students are asked to explain their strategies for some problems as they practice. (DPP items C, G, M, N, X, Z)
- One of the final DPP items in each unit serves as an assessment. (DPP item CC, II)
- One part of the Home Practice provides practice for the facts. (Home Practice Part 2)

Modeling Multiplication. In previous units, students worked with multiplication as a way to solve problems about equal groupings. In this unit, they will use arrays and jumps on the number line to represent multiplication.

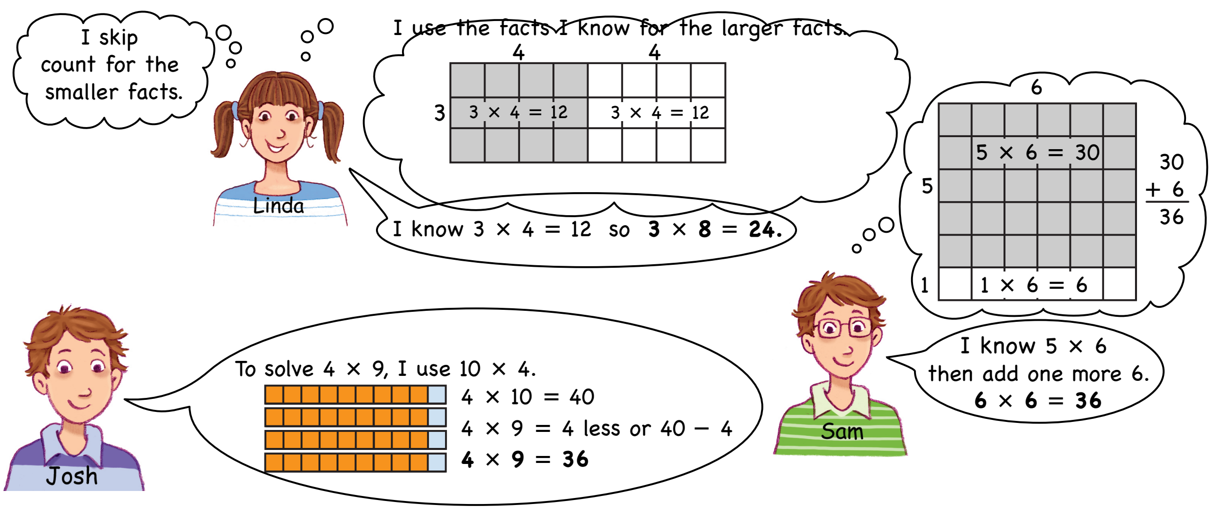

Figure 2: *Some strategies to reason from known multiplication facts*

"Practicing single-digit calculations is essential for developing fluency with them. This practice can occur in many different contexts, including solving word problems. Drill alone does not develop mastery of single-digit combinations. Practice that follows substantial initial experiences that support understanding and emphasize "thinking strategies" has been shown to improve students' achievement with single-digit calculations. This approach allows computation and understanding to develop together and facilitate each other. Explaining how procedures work and examining their benefits, as part of instruction, support retention and yield higher levels of performance.

" … If appropriately delayed, timed tests can benefit some students, but targeted forms of practice, with particular combinations that have yet to be mastered on which efficient procedures can be used, are usually more effective."

From the National Research Council in *Adding It Up: Helping Children Learn Mathematics,* p. 193, 2001.

Constant Hoppers. Using a number line with the same size hops (constant hops) is a representation that connects addition to multiplication and connects division to repeated subtraction. See Figure 4.

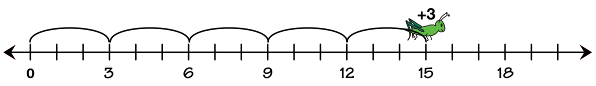

Figure 4: *Number line with constant hopper modeling multiplication*

Arrays. An **array** is an arrangement of elements into a rectangular pattern of (horizontal) **rows** and (vertical) **columns.** For example, a candy box that contains 5 rows with 6 pieces in each row is a 5 × 6 array. One virtue of the array model is that it makes very clear the fact that 5 × 6 = 6 × 5: The box can be rotated 90 degrees to form a 6 × 5 array. Another advantage is that it creates a visual image of a multiplication problem. See Figure 5.

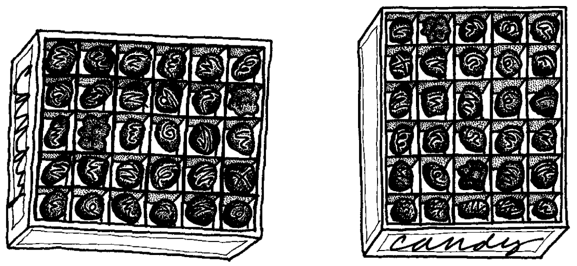

Figure 5: *From left to right, a 5 × 6 array and a 6 × 5 array*

Division Involving Zero. While multiplication involving a zero factor always results in zero as the product (any number times zero is zero), there are two different cases of division involving zero.

1. Zero as the dividend. Example: 0 ÷ 3. When zero is divided by any (non-zero) number, the answer is zero.
   For example, suppose a mother distributes all the cookies in the cookie jar equally among her three children. If there are six cookies, each child will get 6 ÷ 3 = 2 cookies. Similarly, if there are zero cookies, each child will get 0 ÷ 3 cookies. Children will unhappily see that 0 ÷ 3 = 0.
2. Zero is the divisor. Example: 45 ÷ 0. Division by zero is undefined. Here are a few ways to see why division by zero does not make sense:
   a. Division is the inverse of multiplication. Thus, to solve a problem like 45 ÷ 9 = *N*, we ask, "What number times 9 equals 45?" Since 5 × 9 = 45, we know that 45 ÷ 9 = 5. Similarly, to solve the problem 45 ÷ 0, we would ask, "What number times 0 equals 45?" There is no such number, so it is impossible to divide 45 by zero. If we had the same discussion for another number, we would come to the same conclusion, as long as that number is itself not equal to 0. In paragraph c, we will discuss why 0 ÷ 0 is not defined.
   b. Division can also be viewed as repeated subtraction. A simple way to divide by a whole number is to repeatedly subtract that number from the dividend until zero is reached; the quotient is the number of times the divisor is subtracted. For example, to solve 15 ÷ 5 = *N,* we can proceed as follows:

$$\begin{array}{r} 15 \\ \underline{-\ 5} \\ 10 \\ \underline{-\ 5} \\ 5 \\ \underline{-\ 5} \\ 0 \end{array}$$

Since 5 was subtracted three times, we know there are three fives in 15 so that $15 \div 5 = 3$. Now, try the same thing when 0 is the divisor, as in $15 \div 0 = N$:

$$\begin{array}{r} 15 \\ \underline{-\ 0} \\ 15 \\ \underline{-\ 0} \\ 15 \\ \underline{-\ 0} \\ 15 \end{array}$$

It is clear that no progress is being made. If we try to divide by zero by subtracting repeatedly until nothing is left, we will be at it forever. Hence, division by zero is not possible. An example of this can be found in the *Cipher Force!* story in Lesson 8. The silly superhero, Div, tries to fill a roller coaster with 24 students by putting zero students in the first car, zero students in the second car, and so on until all 24 students are on the roller coaster. Even an infinitely long roller coaster would not work!

c. It is tempting to believe that $0 \div 0 = 1$. After all, $7 \div 7 = 1$, $5 \div 5 = 1$, so we could decide that $0 \div 0 = 1$. Using the definition of division in terms of multiplication, $0 \div 0 = N$ means $0 = 0 \times N$. While $N = 1$ does make this number sentence true, so would any other number. Since there is not a unique number that satisfies the condition, $0 \div 0$ is undefined.

There are other ways to see why division by zero does not make sense. All of them require examining the meaning of division.

Math Facts and Mental Math

Subtraction Facts. Students review and are assessed on the subtraction facts in Groups 5–8 to maintain and increase fluency and to learn to apply subtraction strategies to larger numbers. See Mathematics in This Unit in Unit 2 for more about subtraction facts development.

Multiplication Facts. This unit starts a systematic review and assessment of the students' fluency with the multiplication facts. See Figure 1. In this unit students will have opportunities to gain fluency with the multiplication facts for the 5s and 10s. Students should be encouraged to use skip-counting strategies and to reason from facts they know. For example, to solve 5×8 use $10 \times 8 = 80$. Since 5 x 8 is half, $5 \times 8 = 40$.

Algebra in the Early Grades

There are many foundational algebraic ideas explored in this unit. Students explore the multiplication properties of zero and one as well as modeling the distributive property using an array model. Students use stories and examples to reason the outcome of multiplying and dividing by 0 and 1. From this conceptual understanding students can make generalized statements about multiplying and dividing by zero and one.

The array model creates a concrete representation for applying the distributive property of multiplication over addition. This model encourages students to think about decomposing and composing numbers and will be used numerous times in the future as students explore more complex algebraic relationships.

Resources

- Baroody, A.J. "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them." In Teaching Children Mathematics, 13(1), 22-31. National Council of Teachers of Mathematics, Reston, VA, 2006.
- National Research Council. "Developing Proficiency with Whole Numbers." In *Adding It Up: Helping Children Learn Mathematics.* J. Kilpatrick, J. Swafford, and B. Findell, eds. National Academy Press, Washington, DC, 2001.
- National Research Council. "Teaching for Mathematical Proficiency." In *Adding It Up: Helping Children Learn Mathematics.* J. Kilpatrick, J. Swafford, and B. Findell, eds. National Academy Press, Washington, DC, 2001.
- *Principles and Standards for School Mathematics.* National Council of Teachers of Mathematics. Reston, VA, 2000.